alumina of $\epsilon = 9.8$ and height $h = 0.635$ mm. Figs. 7 and 8 show the results of calculations compared with the results of measurements taken by Gronau and Wolff at the University of Duisburg [9]. In this example we took values of $L(w)$ and $C(w)$ for $f = 0$ GHz and $f = 15$ GHz (Figs. 7 and 8).

Very good agreement between the measurements and calculations was also obtained for many other T junctions of different shapes, which are not presented here.

Example 3

We consider a microstrip low-pass filter which has been measured and analyzed by a three-dimensional FDTD method by the authors of [12]. The substrate has $\epsilon_r = 2.2$ and $h = 0.794$ mm. In Figs. 9. and 10 we compare the results published in [12] with those obtained using our 2-D inhomogeneous model. Good agreement again is obtained. Certain discrepancies at high frequencies are most probably due to radiation. Our analysis took about 7 min on a PC-386 working under DOS while the 3-D analysis [12] of the same example was reported to take 8 h on a VAX station 3500.

III. Conclusions

The paper has presented a new two-dimensional model for the analysis of arbitrarily shaped microstrip circuits. The model was checked in the FDTD program prepared by the authors to run on a PC. It was found very useful for investigating new designs of junctions, resonators, and patch couplers. However, it must be admitted that the model is effective only in cases where phenomena that are typically three-dimensional, such as radiation and coupling, can be neglected.

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Accurate Formulas for Efficient Calculation of the Characteristic Impedance of Microstrip Line

K. K. M. Cheng and J. K. A. Everard

***Abstract*—A numerically efficient and accurate method for the derivation of the characteristic impedance of an open microstrip line assuming the quasi-TEM mode of propagation is presented. It is based on the spectral-domain method incorporating functions of rectangular shape for describing the surface charge density distribution on the conductor strip. This gives rise to integrals which can be analytically evaluated. The formulas thus obtained can readily be implemented on a desktop computer. It is found that the discrepancies between the results derived from the proposed method ($N = 3$) and from the substrip method are less than 0.26% through a wide range of w/h ratios and relative permittivity values.**

I. Introduction

A vast amount of literature [1]–[7], [9]–[11] has been published on the numerical computation of the characteristic impedance of microstrip. Wheeler employed an approximate conformal mapping method in the study of microstrip in a mixed dielectric media [2]. Silvester and Farrar [3], [4] treated this problem by the method of moments and dielectric Green's function. Poh *et al.* [5] applied the spectral-domain method to the analysis of microstrip and showed that by careful treatment of the edge singularities of the charge density on the strip, the method can often give rise to accurate results with only a few basis functions. In this paper, a new method based on the spectral-domain approach is presented for determining the characteristic impedance of an open microstrip line. By selecting the rectangular shaped functions as the basis functions, the resulting integrals in the solution can be efficiently evaluated. Furthermore, the number of basis functions required is minimized by searching for the optimum widths of these rectangular shaped functions, which will give the least calculated impedance error. The proposed method is therefore numerically efficient, easy to implement, and highly accurate. For purposes of comparison, results calculated by the substrip method and by our formulas are shown. The extension of this method to the modeling of microstrip with thick strip conductor and covered microstrip is discussed.

II. Method of Analysis

The study of microstrip is carried out under the assumptions that the mode of propagation is quasi-TEM and the line has negligible loss. In this case the characteristic impedance, Z_0, of a microstrip line is given by

$$Z_0 = \frac{1}{v\sqrt{CC_0}} \tag{1}$$

where v is the velocity of light in vacuum, C is the capacitance per unit length of the microstrip shown in Fig. 1, and C_0 is the

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The authors are with the Department of Electronic and Electrical Engineering, King's College, University of London, Strand, London, England WC2R 2LS.

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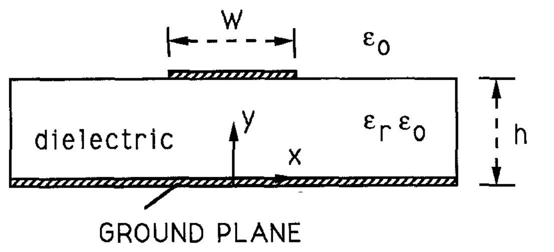

Fig. 1. Cross section of a microstrip line.

capacitance per unit length for the same structure but with $\epsilon_r = 1$. The thickness of the strip is assumed to be zero. For convenience, the potentials of the conductor strip and the ground plane are assumed to be 1 V and 0 V, respectively. The static potential function $\phi(x, y)$ of the microstrip satisfies Poisson's equation:

$$\nabla^2 \phi(x, y) = -\frac{\rho(x)}{\epsilon} \tag{2}$$

and the boundary conditions on the surface of the dielectric material as well as the conductor. Here $\rho(x)$ represents the charge density distribution on the surface of the metal strip. Now we define the Fourier transform as

$$\bar{\phi}(\alpha, y) = \int_{-\infty}^{\infty} \phi(x, y) e^{j\alpha x} \, dx. \tag{3}$$

Taking into account the boundary and continuity conditions, the transformed potential function evaluated at the air–substrate interface $(y = h)$ can be shown [5], [9] to be

$$G(\alpha)\bar{\rho}(\alpha) = \bar{\phi}(\alpha, h) \tag{4}$$

$$G(\alpha) = \frac{1}{\epsilon_0 |\alpha| \{1 + \epsilon_r \coth(|\alpha| h)\}} \tag{5}$$

$$= \frac{1 - e^{-2|\alpha| h}}{\epsilon_0 (1 + \epsilon_r) |\alpha| (1 - k e^{-2|\alpha| h})} \tag{6}$$

and

$$k = \frac{1 - \epsilon_r}{1 + \epsilon_r}. \tag{7}$$

For the infinitely thin strip case, it is well known that both the charge density and the electric field are singular at the edges of such a strip [11]. Of the published methods for finding the charge density distribution, the substrip approximations [3], [4] can be expected to give reasonably good results with sufficiently small subsections. However, this method demands large amounts of computer time and memory. A number of different smooth fitting functions [5]–[7] have been used to approximate these singularities, and these greatly improve the rate and accuracy of computing Z_0. Gladwell and Coen [7] used special functions based on the Chebyshev polynomials in approximating the charge distribution on the strip. However, the numerical quadratures involved in the solutions are far too complicated to use.

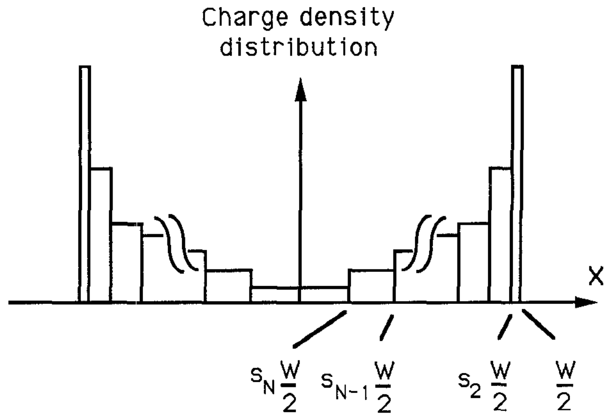

Fig. 2. Charge density distribution model.

III. The New Charge Density Model

It is assumed that the surface charge density on the conductor strip is represented by the distribution model shown in Fig. 2. It consists of N rectangular shaped functions, or, in mathematical form,

$$\rho(x) = \sum_{n=1}^{N} K_n f_{s_n}(x) \tag{8}$$

$$f_{s_n}(x) = \begin{cases} 1, & |x| < \dfrac{s_n W}{2} \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

where $K_1, K_2, \cdots, K_N$ are unknowns yet to be determined. Note that W is the strip width and $0 < s_N < s_{N-1} < \cdots < s_2 < s_1 = 1$. The optimum values of the parameters $s_2, s_3, \cdots, s_N$ will be discussed later. Since the potential on the conductor strip is assumed to be 1 V, the line capacitance of the microstrip is

$$C = W \sum_{n=1}^{N} K_n s_n = W \boldsymbol{K}^T \boldsymbol{S} \tag{10}$$

where $\boldsymbol{K} = [K_1 \; K_2 \cdots K_N]^T$ and $\boldsymbol{S} = [s_1 \; s_2 \cdots s_N]^T$. If we take the Fourier transform of (8), substitute it into (4), and apply Galerkin's method together with Parseval's theorem, we obtain

$$\boldsymbol{A}\boldsymbol{K} = \frac{\pi W}{4} \boldsymbol{S}. \tag{11}$$

Here A is a symmetrical square matrix of dimension N. The elements of the matrix, A_{ij} $(i = 1, \cdots, N;\; j = 1, \cdots, N)$ are given by

$$A_{ij} = \int_0^{\infty} G(\alpha) \frac{\sin\left(\alpha \dfrac{s_i W}{2}\right) \sin\left(\alpha \dfrac{s_j W}{2}\right)}{\alpha^2} \, d\alpha. \tag{12}$$

Note that the above integral may be evaluated analytically as

TABLE I
COMPARISON OF THE PROPOSED FORMULAS AND SUBSTRIP METHOD FOR DIFFERENT ϵ_r AND w/h IN THE CHARACTERISTIC IMPEDANCE CALCULATIONS

w/h \ ϵ_r	6.0		9.6		13.0		28.0	
	PM	SS	PM	SS	PM	SS	PM	SS
0.1	134.78	134.63	109.06	108.94	94.718	94.605	65.612	65.534
0.2	112.58	112.43	91.020	90.891	79.015	78.902	54.699	54.622
0.4	90.482	90.325	73.054	72.927	63.381	63.270	43.835	43.759
0.7	72.892	72.741	58.761	58.638	50.943	50.837	35.194	35.120
1.0	61.987	61.845	49.904	49.789	43.238	43.139	29.843	29.773
2.0	42.376	42.267	34.001	33.913	29.415	29.338	20.249	20.200
4.0	26.503	26.438	21.183	21.131	18.292	18.247	12.555	12.528
10.0	12.745	12.717	10.140	10.118	8.7392	8.7197	5.9808	5.9678

PM—proposed method. SS—substrip method.

follows:

$$A_{ij} = \int_0^\infty \frac{1-e^{-2\alpha h}}{\epsilon_0(1+\epsilon_r)(1-ke^{-2\alpha h})} \frac{\sin\left(\alpha\frac{s_iW}{2}\right)\sin\left(\alpha\frac{s_jW}{2}\right)}{\alpha^3}\,d\alpha$$

$$= \frac{1}{\epsilon_0(1+\epsilon_r)}\int_0^\infty (1-e^{-2\alpha h})(1+ke^{-2\alpha h}+k^2e^{-4\alpha h}+\cdots) \cdot \frac{\sin\left(\alpha\frac{s_iW}{2}\right)\sin\left(\alpha\frac{s_jW}{2}\right)}{\alpha^3}\,d\alpha$$

$$A_{ij} = \frac{1}{\epsilon_0(1+\epsilon_r)}\int_0^\infty (1-e^{-2\alpha h})\frac{\sin\left(\alpha\frac{s_iW}{2}\right)\sin\left(\alpha\frac{s_jW}{2}\right)}{\alpha^3}\,d\alpha + \frac{k}{\epsilon_0(1+\epsilon_r)}\int_0^\infty (1-e^{-2\alpha h})e^{-2\alpha h} \cdot \frac{\sin\left(\alpha\frac{s_iW}{2}\right)\sin\left(\alpha\frac{s_jW}{2}\right)}{\alpha^3}\,d\alpha + \cdots .$$

Each integral in the above expression can be explicitly obtained using the closed-form formulas [8] given in the Appendix. Therefore the coefficients A_{ij} can be rewritten as

$$A_{ij} = \frac{W^2}{4\epsilon_0(1+\epsilon_r)}\left[\{I(p',s_i,s_j)-I(0,s_i,s_j)\} + k\{I(2p',s_i,s_j)-I(p',s_i,s_j)\}+\cdots\right]$$

$$= \frac{W^2}{4\epsilon_0(1+\epsilon_r)}\left\{(1-k)\sum_{n=0}^{\infty}k^nI((n+1)p',s_i,s_j) - I(0,s_i,s_j)\right\} \tag{13}$$

and $p' = 4h/W$. Hence, the line capacitance is given by

$$C = \epsilon_0(1+\epsilon_r)\pi D^TS \tag{14}$$

where the unknown vector $D = [d_1\ d_2 \cdots d_N]^T$ is determined by the following equation:

$$D = B^{-1}S. \tag{15}$$

The elements of the matrix B, B_{ij}, are calculated by the formula

$$B_{ij} = (1-k)\sum_{n=0}^{\infty}k^nI((n+1)p',s_i,s_j) - I(0,s_i,s_j). \tag{16}$$

IV. DISCUSSION

A computer program has been developed for the evaluation of the line capacitance of microstrip based on (14)–(16). The characteristic impedance is then obtained from expression (1). The computed impedance values for the microstrip with different dielectric constants and w/h ratios are shown in Table I. It should be noted that only three rectangular shaped functions have been employed in these calculations. The computer time taken is, of course, greatly dependent upon the number of terms to be retained in the infinite series, and hence on the value of k and the w/h ratios. For k corresponding to relative permittivities in the range 6–28 and $0.1 < w/h < 10$, 30 to 120 terms have been found necessary to ensure less than 0.01% change in capacitance upon doubling the number of terms in the series.

For purposes of comparison, the results for the same transmission line calculated by the substrip method [3], [4] are also included in the table. The standard substrip method can be divided into two parts. The first is the formulation of a suitable Green's function, the second is the solution of the integral equation by writing it in the form of matrix equation and carrying out the matrix inversion numerically. For the substrip method used here the center conductor is divided into 720 subsections. This number is chosen by increasing the number of subsections until the resulting capacitance does not vary by more than 0.01%.

It is quite easy to see that the proposed method ($N = 3$) yields calculated impedance values with maximum error of about 0.26%. The optimum values of s_2 and s_3 used in these calculations are found by computer searching through all the possible values of s_2 and s_3. The goal is to minimize the calculated impedance error with respect to the results obtained by the substrip method given in Table I. The optimum values of s_2 and s_3 are found to be 0.985 and 0.812 respectively. It should be noted that in this case there are only six summations to be evaluated; therefore substantial reductions in the computational storage and time requirements are obtained with these formulas. Clearly, it is possible to achieve higher accuracy in the computed impedance values by increasing the number of basis

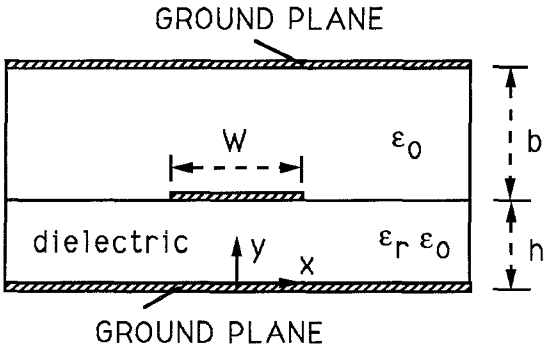

Fig. 3. Covered microstrip line.

functions in (8). For example, we observed that with $N = 10$ the maximum impedance error can be reduced to about 0.06%.

V. Conductor Strip of Finite Thickness

The previous analysis of microstrip is based on the assumption that the strip is of zero thickness. The above formulas may be modified to include the effect of finite strip thickness, say t. Consider two layers of charge situated at $y = h$ and $y = h + t$. Using the same approximations adopted by Yamashita and Mittra [9], the modified expression for $G(\alpha)$ is

$$G(\alpha) = \frac{1 + e^{-|\alpha| t}}{2} \frac{1 - e^{-2|\alpha| h}}{\epsilon_0 (1 + \epsilon_r)|\alpha|(1 - k e^{-2|\alpha| h})}. \tag{17}$$

The new formula for B_{ij} can be shown to be

$$B_{ij} = \frac{1}{2}\Bigg\{(1-k)\sum_{n=0}^{\infty} k^n I((n+1)p', s_i, s_j) - I(0, s_i, s_j) + (1-k)\sum_{n=0}^{\infty} k^n I\left((n+1)p' + \frac{2t}{W}, s_i, s_j\right) - I\left(\frac{2t}{W}, s_i, s_j\right)\Bigg\}. \tag{18}$$

VI. Covered Microstrip

The cross section of a covered microstrip line is shown in Fig. 3. It consists of a conducting strip of zero thickness placed on a dielectric substrate between two parallel ground planes. As b approaches infinity, the open microstrip is obtained. If we consider the solution of the microstrip problem in the quasi-TEM conditions, the function $G(\alpha)$ can be shown [10] as follows:

$$G(\alpha) = \frac{1}{\epsilon_0 |\alpha| \{\coth(|\alpha| b) + \epsilon_r \coth(|\alpha| h)\}} \tag{19}$$

$$= \frac{1}{\epsilon_0 (1 + \epsilon_r)|\alpha|} \sum_{n=0}^{\infty} C_n \{ e^{-2[(n_1+n_3)b + (n_2+n_3)h]|\alpha|} - e^{-2[(n_1+n_3+1)b + (n_2+n_3)h]|\alpha|} + e^{-2[(n_1+n_3+1)b + (n_2+n_3+1)h]|\alpha|} - e^{-2[(n_1+n_3)b + (n_2+n_3+1)h]|\alpha|} \} \tag{20}$$

where

$$C_n = \frac{n!}{n_1! n_2! n_3!}(-1)^{n_1} k^{n_1+n_2}, \qquad n = n_1 + n_2 + n_3.$$

Note that the summation in (20) is to be performed for all combinations of (n_1, n_2, n_3) that give $n = 0, 1, 2, \cdots$ (e.g. for $n = 2$ there are six combinations of n_1, n_2, n_3). Using expression (20), the formulas for the computation of the characteristic impedance of a covered microstrip line can be derived easily based on the procedures described in the previous sections.

VII. Conclusions

A new method for the computation of the characteristic impedance of microstrip has been presented. It has been shown that it is possible to calculate the value of Z_0 to a remarkably close approximation using the formulas derived.

Appendix

$$\int_0^{\infty} (1 - e^{-px}) e^{-npx} \sin(ax)\sin(bx)\frac{dx}{x^3} = I((n+1)p, a, b) - I(np, a, b) \tag{A1}$$

$$I(p, a, b) = \frac{ap}{2}\tan^{-1}\left(\frac{2bp}{p_1}\right) + \frac{bp}{2}\tan^{-1}\left(\frac{2ap}{p_2}\right) - \frac{p_5}{8}\ln(p_3) + \frac{p_6}{8}\ln(p_4) \tag{A2}$$

where

$$p_1 = p^2 + (a+b)(a-b)$$
$$p_2 = p^2 - (a+b)(a-b)$$
$$p_3 = p^2 + (a+b)^2$$
$$p_4 = p^2 + (a-b)^2$$
$$p_5 = p^2 - (a+b)^2$$
$$p_6 = p^2 - (a-b)^2.$$

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